高瞻計畫_振動學課程
Lecture 5: Multiple Degrees of Freedom (I)

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1

Outline

- Introduction to MDOF systems
- Typical MDOF vibration of engineering systems
- Normal mode vibration analysis
- Essential linear algebra
- Natural frequencies and natural modes
- System couplings
- Forced vibration analysis
- Simple problem
- Demonstrations

2

Part I. Introduction to MDOF systems

Introduction

- Multiple degree of freedom vibration
  - Closer to the real scenario
    - Human body, car suspension, turbine, building etc.
  - Mass → mass matrix; spring constant → stiffness matrix
  - Procedure
    - Convert engineering system to MDOF m-b-k model
    - Obtain the equations
    - Solve the equation using linear algebra
    - Solve the equation using ODE
    - Interpret the result and perform design recommendation

Introductory Example: Car Safety Design

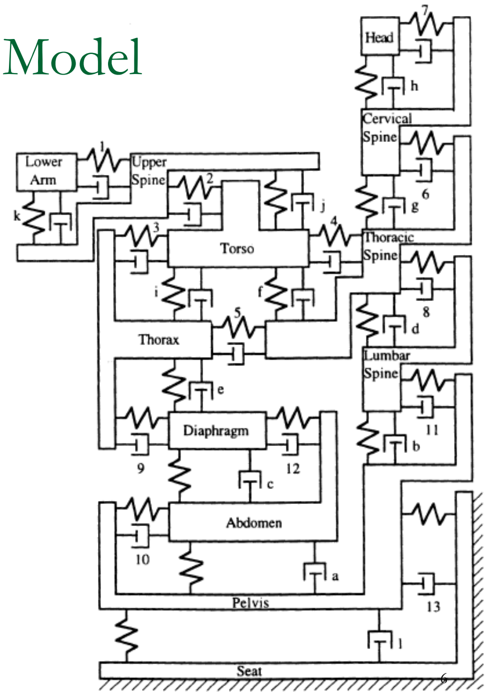
Human Vibration Model

$$[m]\ \ddot{\vec{x}} + [c]\ \dot{\vec{x}} + [k]\ \vec{x} = \vec{F}$$

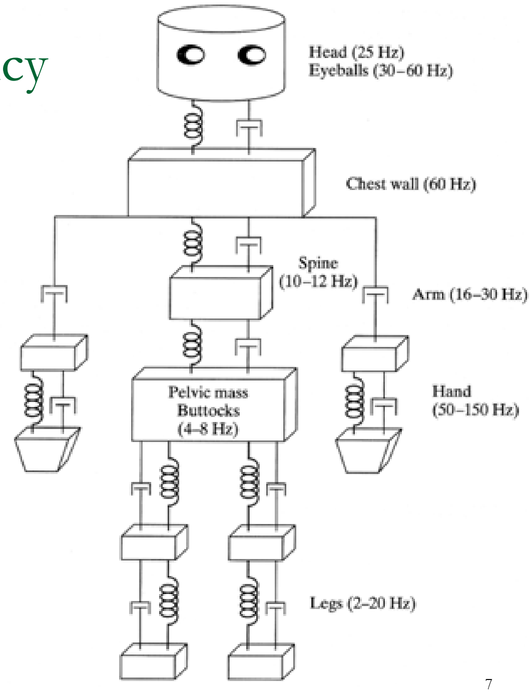
$$[m] = \begin{bmatrix} m_{11} & m_{12} & m_{13} & \cdots & m_{1n} \\ m_{12} & m_{22} & m_{23} & \cdots & m_{2n} \\ \cdot & & & & \\ \cdot & & & & \\ \cdot & & & & \\ m_{1n} & m_{2n} & m_{3n} & \cdots & m_{nn} \end{bmatrix}$$

$$[c] = \begin{bmatrix} c_{11} & c_{12} & c_{13} & \cdots & c_{1n} \\ c_{12} & c_{22} & c_{23} & \cdots & c_{2n} \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ c_{1n} & c_{2n} & c_{3n} & \cdots & c_{nn} \end{bmatrix}$$

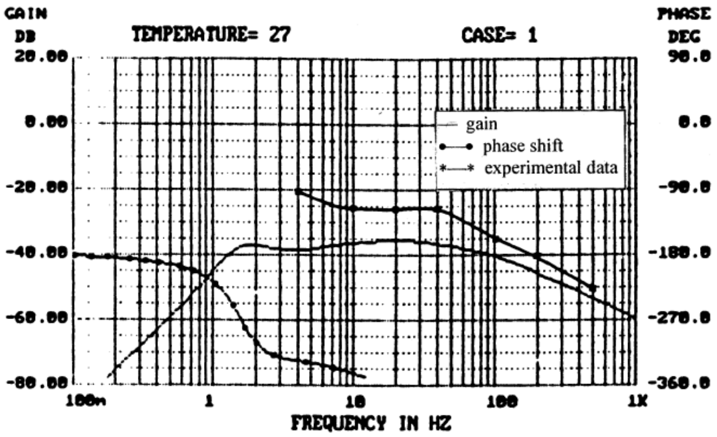
$$[k] = \begin{bmatrix} k_{11} & k_{12} & k_{13} & \cdots & k_{1n} \\ k_{12} & k_{22} & k_{23} & \cdots & k_{2n} \\ \cdot & & & & \\ \cdot & & & & \\ \cdot & & & & \\ k_{1n} & k_{2n} & k_{3n} & \cdots & k_{nn} \end{bmatrix}$$



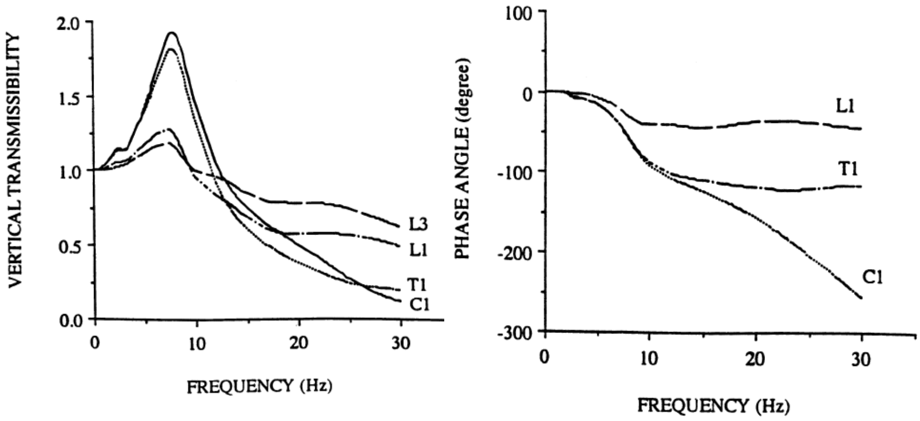
Vibration Frequency Sensitivity of Different Part



Evaluate System Characteristics

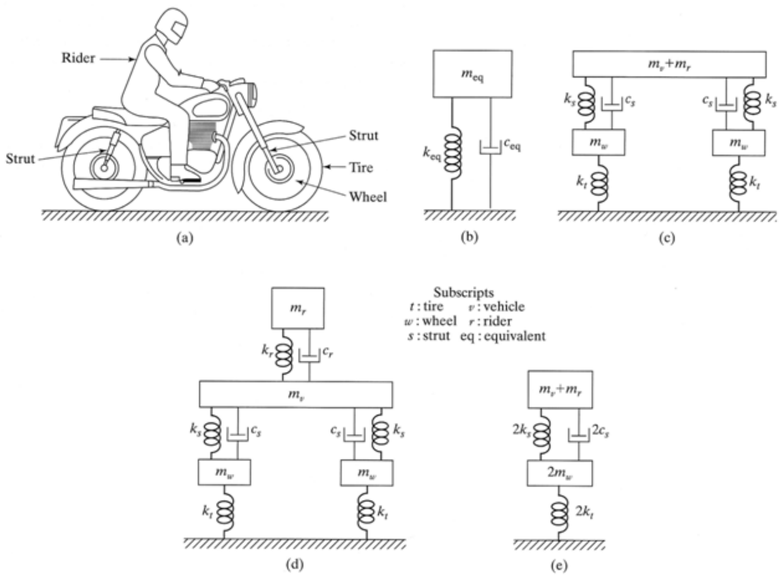


Influence of Car Seats on Vibration



9

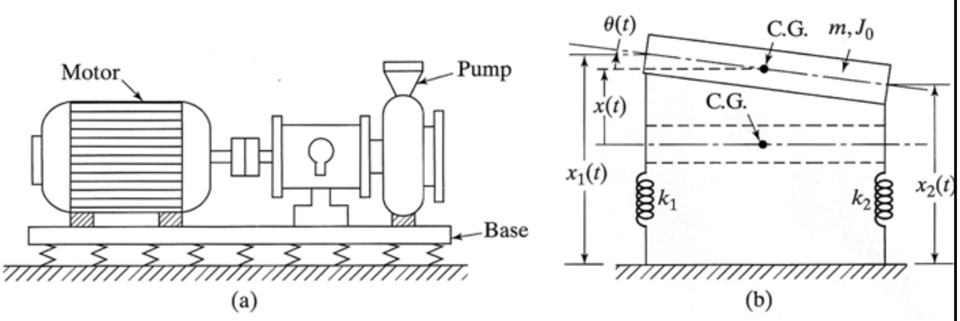
Motorcycle



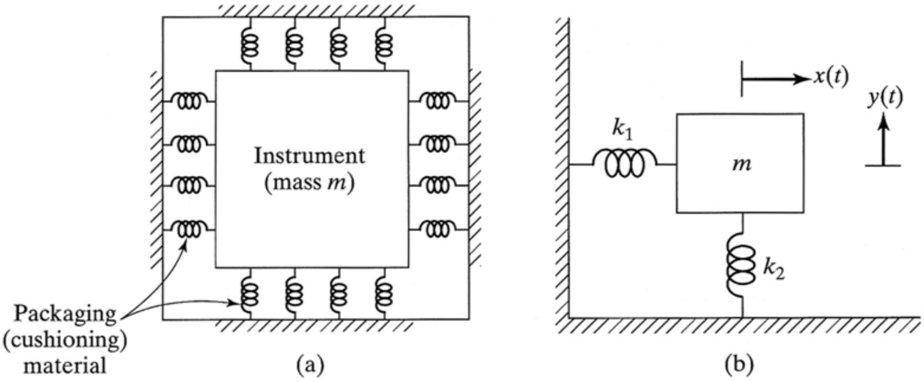
10

Part II. MDOF vibration of Engineering Systems

Motor-Pump on Elastic Foundation

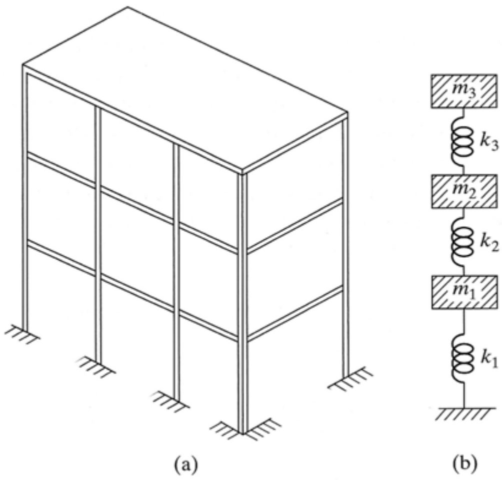


Packaging of an Instrument



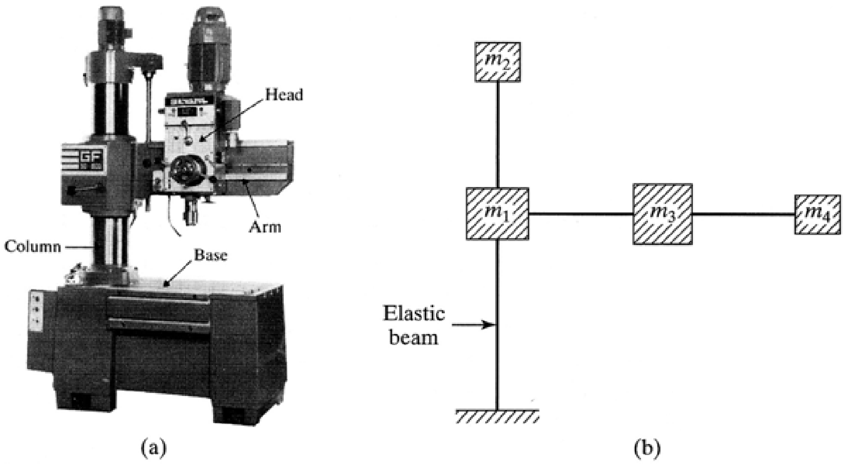
13

Three-story Building



14

A CNC Machine



Horizontal Milling Machine

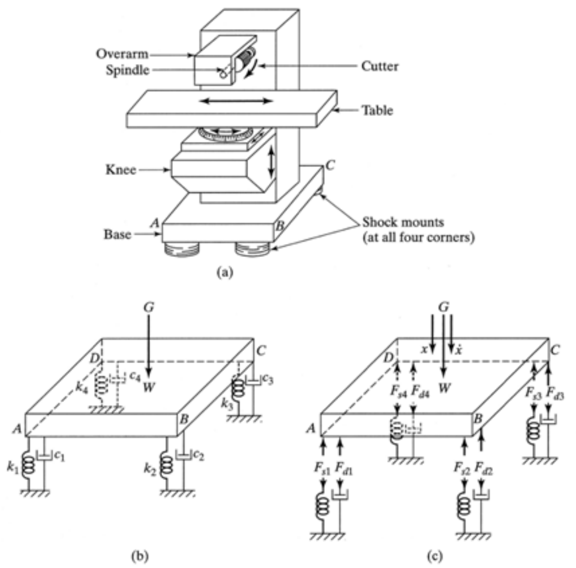
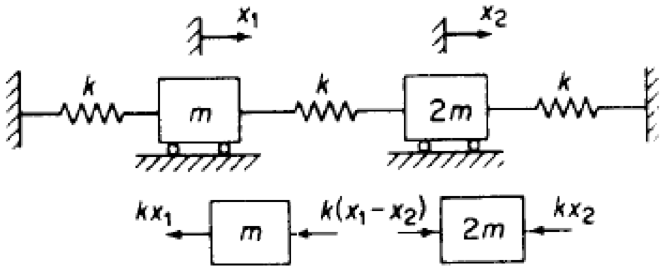


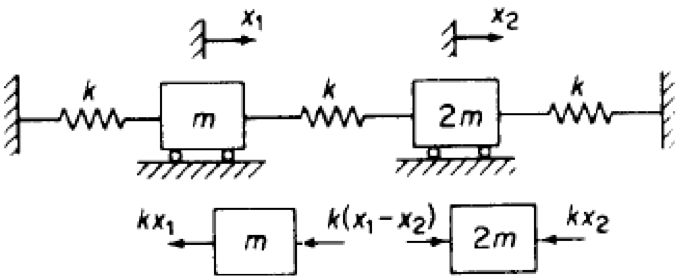
FIGURE 1.37 Horizontal milling machine.

Typical MDOF Vibration Model



Part III. Normal Mode Analysis

Example Problem



Equations of Motion

$$x_1 = A_1 e^{i\omega t}$$

$$x_2 = A_2 e^{i\omega t}$$

$$(2k - \omega^2 m)A_1 - kA_2 = 0$$

$$-kA_1 + (2k - 2\omega^2 m)A_2 = 0$$

$$\begin{vmatrix} (2k - \omega^2 m) & -k \\ -k & (2k - 2\omega^2 m) \end{vmatrix} = 0$$

Letting $\omega^2 = \lambda$.

$$\lambda^2 - \left(3\frac{k}{m}\right)\lambda + \frac{3}{2}\left(\frac{k}{m}\right)^2 = 0$$

Solution

$$\lambda_1 = \left(\frac{3}{2} - \frac{1}{2}\sqrt{3}\right)\frac{k}{m} = 0.634\frac{k}{m}$$

$$\lambda_2 = \left(\frac{3}{2} + \frac{1}{2}\sqrt{3}\right)\frac{k}{m} = 2.366\frac{k}{m}$$

$$\omega_1 = \lambda_1^{1/2} = \sqrt{0.634\frac{k}{m}}$$

$$\omega_2 = \lambda_2^{1/2} = \sqrt{2.366\frac{k}{m}}$$

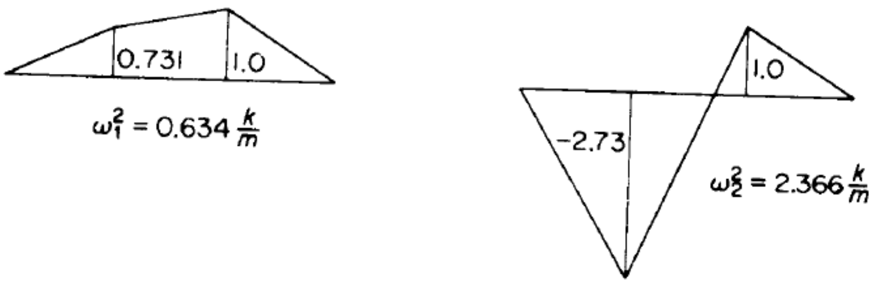
Solution

$$\left(\frac{A_1}{A_2}\right)^{(1)} = \frac{k}{2k - \omega_1^2 m} = \frac{1}{2 - 0.634} = 0.731$$

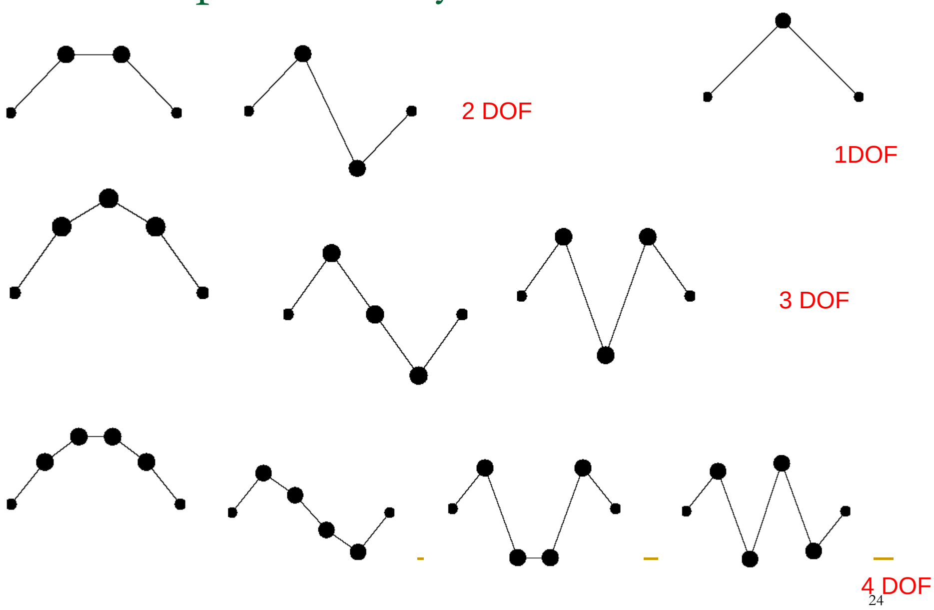
$$\left(\frac{A_1}{A_2}\right)^{(2)} = \frac{k}{2k - \omega_2^2 m} = \frac{1}{2 - 2.366} = -2.73$$

$$\phi_1(x) = \begin{Bmatrix} 0.731 \\ 1.00 \end{Bmatrix} \qquad \phi_2(x) = \begin{Bmatrix} -2.73 \\ 1.00 \end{Bmatrix}$$

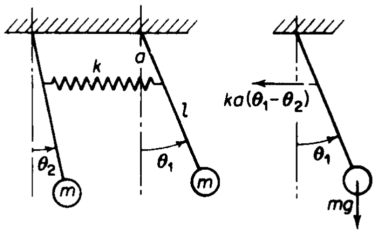
Mode Shapes



Multiple DOF Systems



Example: Coupled Pendulum



$$ml^2\ddot{\theta}_1 = -mgl\theta_1 - ka^2(\theta_1 - \theta_2)$$

$$ml^2\ddot{\theta}_2 = -mgl\theta_2 + ka^2(\theta_1 - \theta_2)$$

Continue

$$\theta_1 = A_1 \cos \omega t$$

$$\theta_2 = A_2 \cos \omega t$$

$$\omega_1 = \sqrt{\frac{g}{l}} \qquad \omega_2 = \sqrt{\frac{g}{l} + 2\frac{k}{m}\frac{a^2}{l^2}}$$

$$\left(\frac{A_1}{A_2}\right)^{(1)} = 1.0 \qquad \left(\frac{A_1}{A_2}\right)^{(2)} = -1.0$$

if the initial conditions are $\theta_1(0) = A$ and $\theta_2(0) = 0$,

$$\theta_1(t) = \tfrac{1}{2}A \cos \omega_1 t + \tfrac{1}{2}A \cos \omega_2 t$$

$$\theta_2(t) = \tfrac{1}{2}A \cos \omega_1 t - \tfrac{1}{2}A \cos \omega_2 t$$

Continue (Beating)

$$\theta_1(t) = A\cos\left(\frac{\omega_1-\omega_2}{2}\right)t\cos\left(\frac{\omega_1+\omega_2}{2}\right)t$$

$$\theta_2(t) = -A\sin\left(\frac{\omega_1-\omega_2}{2}\right)t\sin\left(\frac{\omega_1+\omega_2}{2}\right)t$$

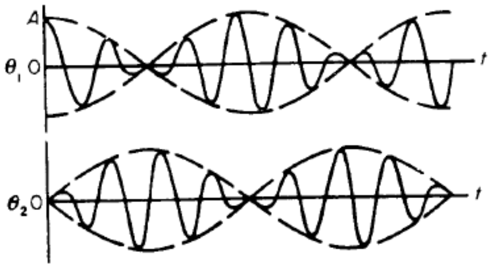
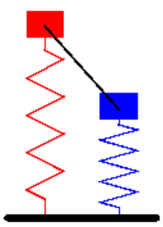
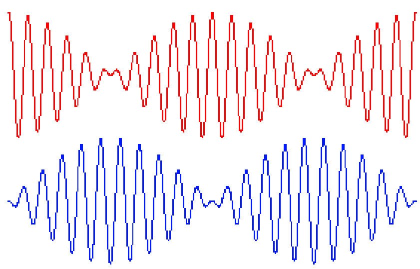


Figure 5.1-6. Exchange of energy between pendulums.

Beating: Visualization





Part IV. Essential Linear Algebra

Eigenvalues / Eigenvectors

- Matrix Algebra: Eigenvalue problem
  - Solve AX=λX to find specific λ
  - λs satisfy the problem is the eigenvalues
  - For each λ, the corresponding X is the eigenvector
- For vibration problem

$$[K - \omega^2 M]X = 0$$

  - ω: natural frequency

Matrix Diagnalization / System Decouple

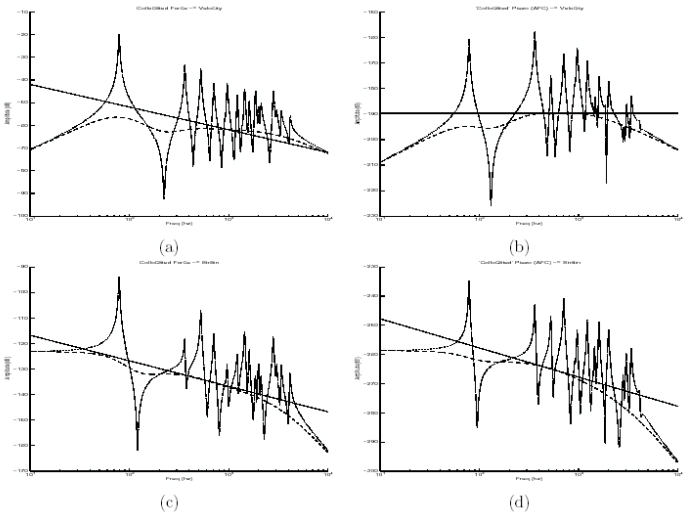
- Modal matrix P
  - A matrix form by all eigenvector (aka modal matrix)

$$[M]\ddot{\underline{x}}+[K]\underline{x}=0$$
$$[M_D]=P^{-1}MP$$
$$[K_D]=P^{-1}KP$$
$$[M_D]\ddot{\underline{x}}+[K_D]\underline{x}=0$$

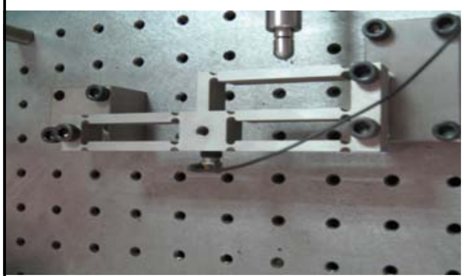
  - Where M_D and K_D are diagonized mass and stiffness matrix

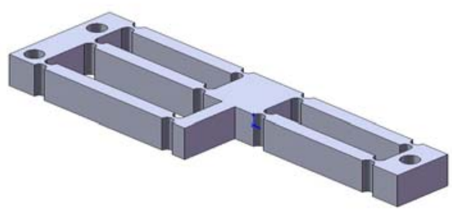
Part V. Natural Frequencies/Modes

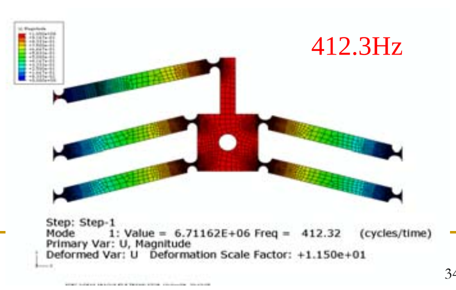
Natural Frequencies



Natural Modes







Part VI. System Coupling

Introduction

- Coupling depends on the seelction of coordinates
- For undamped system, it is always possible to find a particular coordinate set to decouple the system
  - Principal direction, or normal coordinate
- For damped system, in general, it cannot decouple the system
  - Except “proportional” damping

A 2-DOF Example

$$\begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

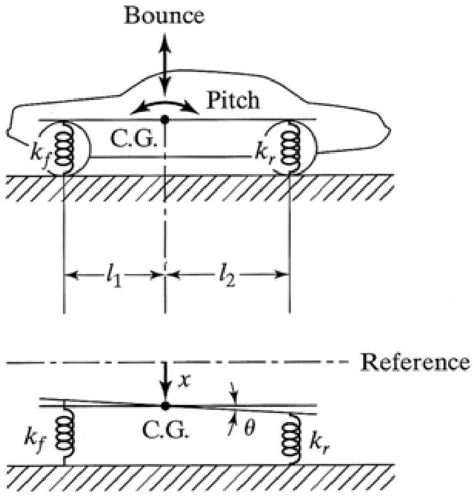
System with damping

$$\begin{bmatrix} m_{11} & 0 \\ 0 & m_{22} \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} \begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{Bmatrix} + \begin{bmatrix} k_{11} & 0 \\ 0 & k_{22} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 0 \end{Bmatrix}$$

If $C_{12}=C_{21}=0$ (Proportional damping), then the system is uncoupled

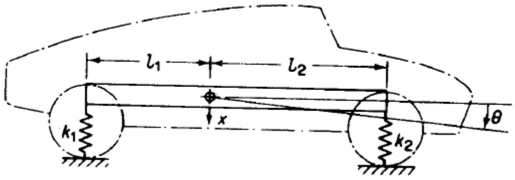
37

Car Suspension Example



38

Static Coupling



Mass matrix decoupled, stiffness matrix coupled

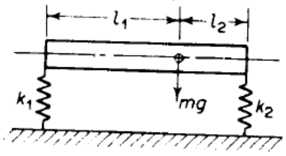


Figure 5.2-1.

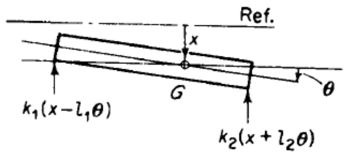


Figure 5.2-2. Coordinates leading to static coupling.

$$\begin{bmatrix} m & 0 \\ 0 & J \end{bmatrix} \begin{Bmatrix} \ddot{x} \\ \ddot{\theta} \end{Bmatrix} + \begin{bmatrix} (k_1 + k_2) & (k_2 l_2 - k_1 l_1) \\ (k_2 l_2 - k_1 l_1) & (k_1 l_1^2 + k_2 l_2^2) \end{bmatrix} \begin{Bmatrix} x \\ \theta \end{Bmatrix} = \begin{Bmatrix} 0 \end{Bmatrix}$$

If $k_1 l_1 = k_2 l_2$, the coupling disappears, and we obtain uncoupled x and θ vibrations.

Dynamic Coupling

Mass matrix coupled, stiffness matrix decoupled

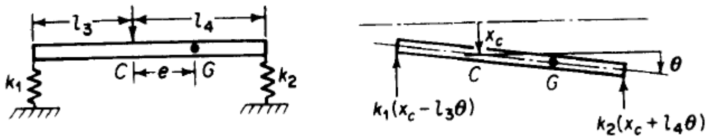


Figure 5.2-3. Coordinates leading to dynamic coupling.

$$\begin{bmatrix} m & me \\ me & J_c \end{bmatrix} \begin{Bmatrix} \ddot{x}_c \\ \ddot{\theta} \end{Bmatrix} + \begin{bmatrix} (k_1 + k_2) & 0 \\ 0 & (k_1 l_3^2 + k_2 l_4^2) \end{bmatrix} \begin{Bmatrix} x_c \\ \theta \end{Bmatrix} = \begin{Bmatrix} 0 \end{Bmatrix}$$

40

Static and Dynamic Coupling

Mass matrix coupled, stiffness matrix coupled

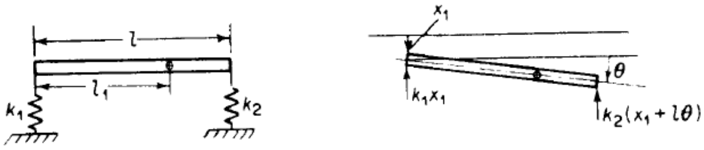


Figure 5.2-4. Coordinates leading to static and dynamic coupling.

$$\begin{bmatrix} m & ml_1 \\ ml_1 & J_1 \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{\theta} \end{Bmatrix} + \begin{bmatrix} (k_1 + k_2) & k_2 l \\ k_2 l & k_2 l^2 \end{bmatrix} \begin{Bmatrix} x_1 \\ \theta \end{Bmatrix} = \begin{Bmatrix} 0 \end{Bmatrix}$$

41

Part VII. Forced Vibration Analysis

42

Introductory Example

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ 0 \end{Bmatrix} \sin \omega t$$

Let $\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} \sin \omega t$

$$\Rightarrow \begin{bmatrix} (k_{11} - m_1\omega^2) & k_{12} \\ k_{21} & (k_{22} - m_2\omega^2) \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ 0 \end{Bmatrix}$$

Or $[Z(\omega)] \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ 0 \end{Bmatrix}$

43

Cont'd

$$\begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} = [Z(\omega)]^{-1} \begin{Bmatrix} F_1 \\ 0 \end{Bmatrix} = \frac{adj[Z(\omega)] \begin{Bmatrix} F_1 \\ 0 \end{Bmatrix}}{|Z(\omega)|}$$

Or $\begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} = \frac{1}{|Z(\omega)|} \begin{bmatrix} (k_{22} - m_2\omega^2) & -k_{12} \\ -k_{21} & (k_{11} - m_1\omega^2) \end{bmatrix} \begin{Bmatrix} F \\ 0 \end{Bmatrix}$

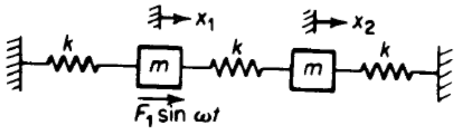
Or

$$X_1 = \frac{(k_{22} - m_2\omega^2)F}{m_1 m_2(\omega_1^2 - \omega^2)(\omega_2^2 - \omega^2)}$$

$$X_2 = \frac{-k_{12}F}{m_1 m_2(\omega_1^2 - \omega^2)(\omega_2^2 - \omega^2)}$$

44

Example



$$\begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} 2k & -k \\ -k & 2k \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ 0 \end{Bmatrix} \sin \omega t$$

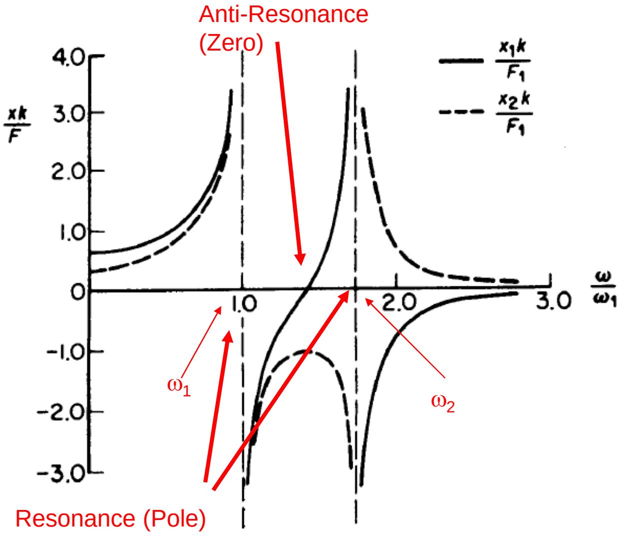
$$X_1 = \frac{(2k - m\omega^2)F_1}{m^2(\omega_1^2 - \omega^2)(\omega_2^2 - \omega^2)}$$

$$X_2 = \frac{kF_1}{m^2(\omega_1^2 - \omega^2)(\omega_2^2 - \omega^2)}$$

$$\omega_1^2 = k/m;$$

$$\omega_2^2 = 3k/m.$$

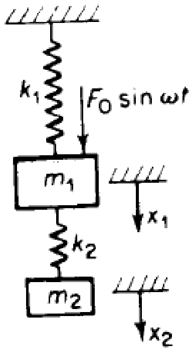
Response



Vibration Absorber

m_1, k_1

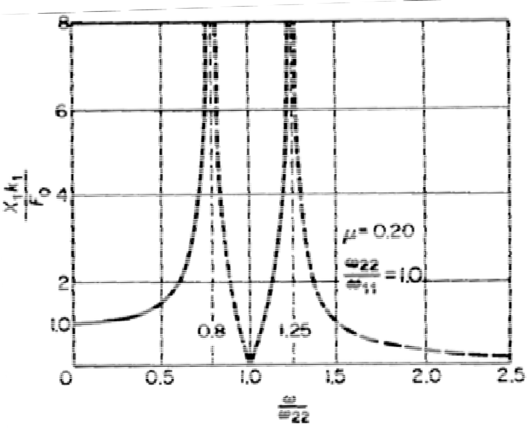
m_2, k_2



Adding extra mass/spring to reduce resonance vibration

47

Response

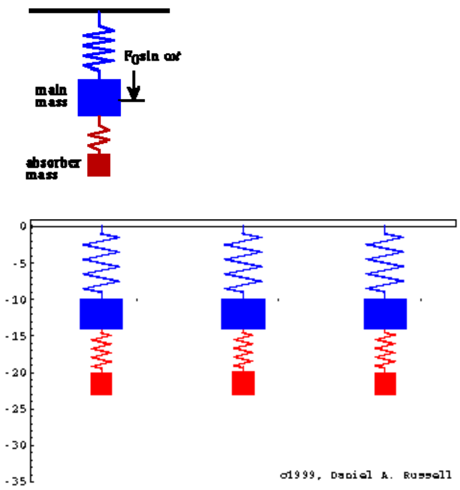


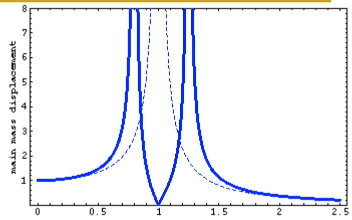
$$\omega_{11}^2 = \frac{k_1}{m_1} \qquad \omega_{22}^2 = \frac{k_2}{m_2}$$

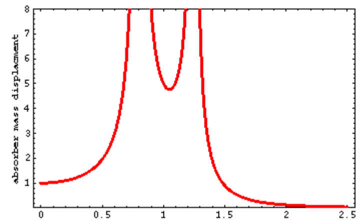
$$\frac{X_1 k_1}{F_0} = \frac{\left[1 - \left(\frac{\omega}{\omega_{22}}\right)^2\right]}{\left[1 + \frac{k_2}{k_1} - \left(\frac{\omega}{\omega_{11}}\right)^2\right]\left[1 - \left(\frac{\omega}{\omega_{22}}\right)^2\right] - \frac{k_2}{k_1}}$$

48

Dynamic Absorber

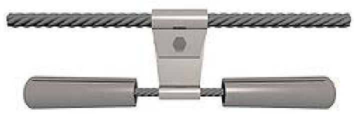






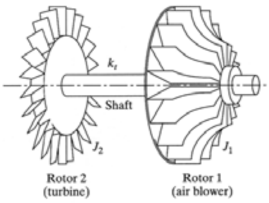
Example: Stockbridge Damper

- a tuned mass damper used to suppress wind-induced vibrations on taut cables, such as overhead power lines
- consists of two masses at the ends of a short length of cable or flexible rod
- damper is designed to dissipate the energy of oscillations in the main cable to an acceptable level.
- Aka "**dog-bone damper**"



Semi-Definite Systems

- Vibration system containing rigid-body mode
  - Rigid body mode, vibration frequency =0
- Examples

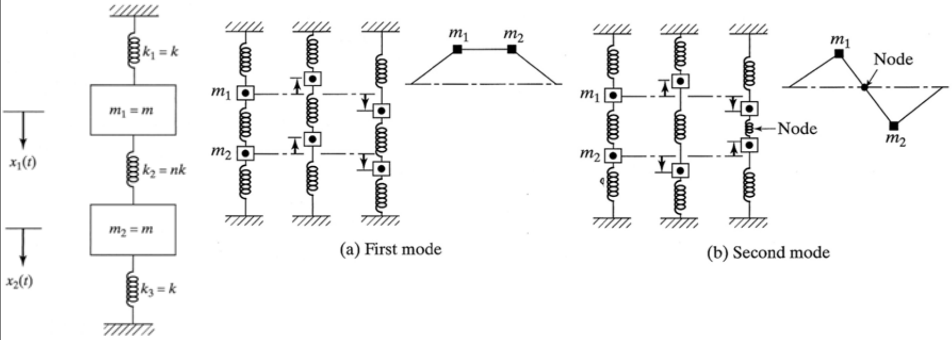


51

Part VIII. Simple Problems

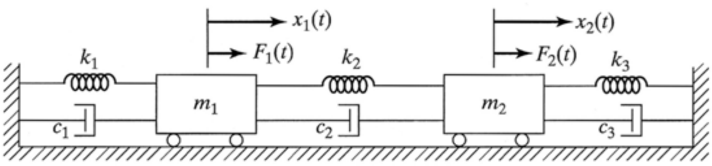
52

Problem 1. Frequencies of Mass-Spring System (Rao. 5.1)



Find the natural frequencies and mode shapes of a spring-mass system, shown in Fig. 5.4, which is constrained to move in the vertical direction only. Take $n = 1$.

Problem 2. Free Vibration of a 2-DOF system (Rao. 5.3)



Find the free vibration response of the system shown in Fig. 5.3(a) with $k_1 = 30$, $k_2 = 5$, $k_3 = 0$, $m_1 = 10$, $m_2 = 1$ and $c_1 = c_2 = c_3 = 0$ for the initial conditions $x_1(0) = 1$, $\dot{x}_1(0) = x_2(0) = \dot{x}_2(0) = 0$.

Problem 3. Torsional Vibration (Rao. 5.4)

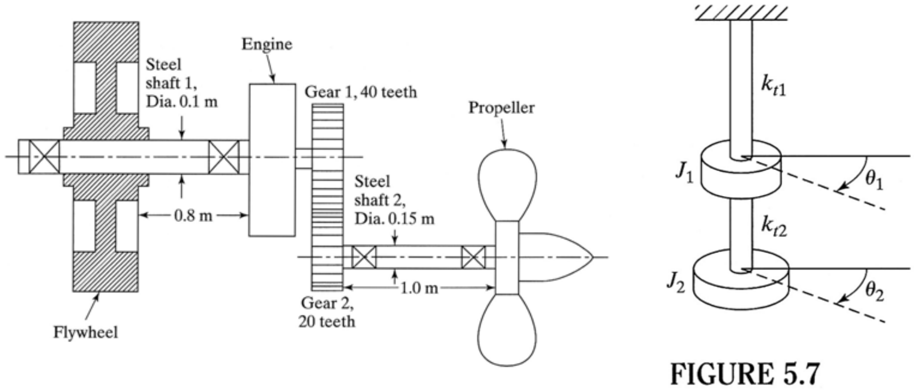
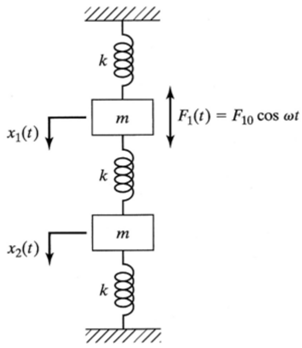


FIGURE 5.7
Torsional system.

Find the natural frequencies and mode shapes for the torsional system shown in Fig. 5.7 for $J_1 = J_0$, $J_2 = 2J_0$, and $k_{t1} = k_{t2} = k_t$.

Problem 4. Steady State Response of a 2-DOF System (Rao. 5.8)



Find the steady-state response of the system shown in Fig. 5.13 when the mass m_1 is excited by the force $F_1(t) = F_{10} \cos \omega t$. Also, plot its frequency response curve.

Part IX. Demonstrations